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Location choice and risk attitude of a decision maker ☆

Oded Berman [a], Nima Sanajian [a,*], Jiamin Wang [b]

[a] University of Toronto, Joseph L. Rotman School of Management, 105 St. George Street, Toronto, Ontario, Canada M5S 3E6
[b] College of Management, Long Island University Post Campus, 720 Northern Blvd, Brookville, New York 11548, USA

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ABSTRACT

In this paper we study the effect of a decision maker's risk attitude on the median and center problems, two well-known location problems, with uncertain demand in the mean–variance framework. We provide a mathematical programming formulation for both problems in the form of quadratic programming and develop solution procedures. In particular, we consider the vertex and absolute median problems separately, and identify a dominant set for the center problem. Glover's linearization method is applied to solve the vertex median problem. We also develop a branch and bound algorithm and a heuristic as the linearization technique takes too long for the vertex median problem on large networks. A computational experiment is conducted to compare the performance of the algorithms. We demonstrate the importance of taking into account the volatility and correlation structure when a location decision is made. The closest assignment property is also discussed for these location problems under the mean–variance objective.

1. Introduction

Decisions to locate facilities such as plants, warehouses and shopping malls are very important, and are often classified as strategic decisions [32,26,16,27]. They usually result in significant fixed costs and more importantly they have considerable impacts on growth prospects of a firm. Moreover, relocating facilities is usually not easy and very costly. As a result, location decisions are made carefully as the executives are aware of their significant economical importance.

To our knowledge, there has been no literature on managerial perceptions of risk specifically for location decision-making problems. However, many studies of risk taking by business executives and managers have attested the importance of risk assessment and management to decision making from the managerial perspective [6]. Most managers interviewed in these studies depicted themselves as risk averse or risk seeking. It has been inferred that their risk attitudes could be attributed to cultural, organizational, occupational and individual differences. Given the substantial impact of a facility location decision, it is arguable that the decision maker may not always be risk neutral, a common assumption in the facility location literature.

1.1. Risk analysis in facility location

It is assumed that the decision maker is risk neutral in all the early and much of the recent facility location literature, in particular, on the median and center problems. However, there are studies that introduce the notion of volatility to these classical location problems. Table 1 summarizes the risk analysis measures used in these studies.

The probability-related measure approach seeks to maximize the probability to achieve a target level of distance or coverage. Value-at-risk (VaR) and conditional value-at-risk (CVaR) are popular measures of risk in finance. β-VaR and β-CVaR at a probability level β are defined, respectively, as the β-quantile of a random loss (or cost) and the conditional expected loss (or cost) exceeding β-VaR [28].

The mean–variance theory [24] is classical in financial portfolio management that makes a trade-off between the mean return and the associated risk. A mean–variance optimization model is to maximize the mean–variance objective function

$$U(Y) = E(Y) - \lambda Var(Y), \tag{1}$$

where Y is a random payoff with mean $E(Y)$ and variance $Var(Y)$, $U(Y)$ is the decision maker's utility, and λ is a risk attitude coefficient. Note that the decision maker is risk averse, risk neutral and risk seeking when $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$, respectively. The magnitude of λ reflects the degree of the decision maker's risk attitude. It was shown that the mean–variance objective is consistent with a quadratic expected utility function [17,30,29,9].

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* Corresponding author.
E-mail addresses: berman@rotman.utoronto.ca (O. Berman), nima.sanajian06@rotman.utoronto.ca (N. Sanajian), jiamin.wang@liu.edu (J. Wang).

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Table 1
Selected literature on risk analysis in the median and center problems.

Risk measure	Single-facility problems	Multiple-facility problems
Probability measure	Frank [10,11]	
	Berman et al. [3]	
Variance	Frank [10]	
Mean–variance		Jucker and Carlson [20]
		Hodder and Jucker [18]
Value-at-risk	Wang [34,35]	Wagner et al. [33]

The mean–variance approach has been criticized for taking into account both the favorable and unfavorable deviations of the random payoff Y from the mean $E(Y)$ in its risk measure, namely the variance $Var(Y)$. As punishing desirable fluctuations when the probability distribution of Y is asymmetric may lower the mean payoff, alternative measures that consider the downside risk only have been proposed [25]. However, the studies by Grootveld and Hallerbach [15], and Choi and Chiu [8] suggested that the mean–variance approach and the mean-downside-risk approaches tend to return similar results in most cases. Grootveld and Hallerbach [15] also pointed out that downside-risk measures were much more prone to estimation risk than the variance.

Knowing the controversy over the mean–variance approach, we nonetheless adopt Eq. (1) as the optimization criterion in the current study for the reasons stated below:

- The mean–variance approach is applicable to explore the trade-off between the mean and variance of the random payoff Y for any stochastic optimization problem, including the location analysis problem.
- Using variance to quantify risk is intuitive, the mean–variance model can be applied by decision makers of different risk attitudes (risk neutral, risk averse and risk seeking), and the mean–variance approach requires only the first two moments of the random variable Y. On the contrary, alternative risk measures such as VaR and CVaR usually reflect the downside risk-averse behavior only, and entail the knowledge of the probability distribution of the random payoff Y.
- Unlike the mean–variance optimization model with a constraint to bound variance from above, the mean–variance objective (Eq. (1)) does not exclude solutions with a high mean payoff and high variability from consideration. Therefore, we would not expect that the negative impact the approach's flaw has on solution quality be significant as long as the risk attitude coefficient λ is not too large.

As a remark, we realize that there are controversies over the pros and cons of various risk measures and believe that a comparative study on these measures for location problems would greatly facilitate the application of stochastic location analysis models.

The mean–variance objective was used by Jucker and Carlson [20] and Hodder and Jucker [18] to study the uncapacitated plant-location problem with uncertain prices and demand. Hodder and Jucker's problem is to some extent similar to the version of the p-median problem to be studied in this paper. A major difference is that they chose a very specific correlation structure for the prices charged by facilities (which were the random variables in their model) whereas we use a general correlation structure for the random demand weights. Consequently, our optimization problem is much harder than theirs.

In Wagner et al. [33], the uncapacitated plant-location problem was studied with an objective to maximize the VaR of a future profit. Under the normality assumption the objective function was converted into the mean–variance framework and a nonlinear integer programming model was solved. The algorithmic approach proposed by Wagner et al. [33] works only for a risk-averse decision maker. The authors reported a computational experiment on small networks only for the vertex version of the model, where facilities can be located on the nodal points only. Different from their study, in this paper we discuss both the absolute (facilities can be located anywhere on the network) and vertex p-median problems, and conduct an extensive computational study on networks of various sizes. We also develop a motivating example to show that when the mean value of Y is non-decreasing in the variance the decision maker may have a reason to be risk seeking, and present algorithms that are not restricted by the decision maker's risk attitude.

1.2. Outline of the study

In this paper we consider the decision maker's risk attitude and investigate how it can change the optimal solutions to the median and center location problems with uncertain demand weights that were well studied under the assumption of risk-neutrality. Our main objective is to show how the decision maker's risk attitude can affect the optimal facility locations. We also try to shed some light on the important role that demand variability and correlation structure play to choose optimal risk-averse or risk-seeking locations.

Under the mean–variance objective, each optimization problem is formulated in the form of a quadratic programming model. We discuss how we can solve the problems using linearization techniques, which have been shown to be quite effective for problems with quadratic objective function [7]. Specifically, Glover's [14] linearization method is adopted for the vertex median problem. We also develop a branch and bound algorithm and a vertex substitution heuristic for the problem because our computational experience suggests that the linearization method is not always efficient for large networks.

The rest of the paper is organized as follows. In Section 2, we discuss the mean–variance objective, and in Sections 3 and 4 we analyze the median and center problems with uncertain demand under the mean–variance objective. In Section 5, we provide insights on optimal locations under different risk preferences and changing parameters. In Section 6, a computational experiment is reported to compare the performance of the algorithms developed for the models. Finally, we provide a brief summary and outlook in Section 7.

2. The mean–variance objective

Let $G=(N,L)$ be an undirected network with a set of nodes N ($|N|=n$) and a set of links L. Let x denote both the location of point x on link (a,b) and the distance between this point and the left end node a. The shortest distance between any two points x and y located somewhere in G is denoted by $d(x,y)=d(y,x)$. To uniquely define a link $(a,b)\in L$, it is required that the index of the left end node a is smaller than that of the right end node b. We further require that the length of each link (a,b), denoted by l_{ab}, is equal to the shortest distance between nodes a and b.

Given a set of p ($p<n$) points $\mathbf{X}_p=(x_{(1)},x_{(2)},\cdots,x_{(p)})$, let $d(x,\mathbf{X}_p)=\min_{1\le j\le p}\{d(x,x_{(j)})\}$. Demand is assumed to originate from the nodes of G only. The demand weight at node i, h_i, is random with mean μ_i and standard deviation σ_i. Random variables h_i and h_k may be correlated with a correlation coefficient ρ_{ik}.

We study the median and center problems with random demand weights under the mean–variance objective (Eq. (1)), for which the random payoff Y will be, respectively, defined for either problem. In Section 5 we will show that the impact of λ on optimal locations also depends on the values of $E(Y)$ and $Var(Y)$. In the current study we attempt to develop general models and algorithms that are independent of $\lambda \neq 0$. The reader is referred to Jucker and Carlson [20] for a summary of four suggested methods to select λ.

Example 1. To motivate our study, consider a 1-median problem with uncertain demand weights on a segment of z units long. For simplicity, we assume that the two random demand weights associated with the two end nodes 1 and 2 of the segment are independent.

The deterministic 1-median problem is to locate a single facility so as to minimize the total weighted distance from demand to the facility. Let x be the location site of a single facility. When the demand weights are uncertain, the random payoff Y for the 1-median problem is formulated as $Y(x) = -h_1 d(1,x) - h_2 d(2,x)$.

Note that $-Y(x) = h_1 d(1,x) + h_2 d(2,x)$ is the total weighted distance from demand to the facility located at point x. It is easy to derive

$$E(-Y(x)) = \mu_1 x + \mu_2 (z-x),$$

and

$$Var(-Y(x)) = \sigma_1^2 x^2 + \sigma_2^2 (z-x)^2. \tag{2}$$

Let $u(x) = E(-Y(x))$ and $v(x) = Var(-Y(x))$. We have

$$\frac{du(x)}{dx} = \mu_1 - \mu_2,$$

$$\frac{dv(x)}{dx} = 2(\sigma_1^2 + \sigma_2^2)x - 2\sigma_2^2 z,$$

and

$$\frac{dv(x)}{du(x)} = \frac{2(\sigma_1^2 + \sigma_2^2)x - 2\sigma_2^2 z}{\mu_1 - \mu_2}.$$

It follows that the mean and variance of the total weighted distance are positively correlated, negatively correlated, and not correlated if $x > \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} z$ and $\mu_1 > \mu_2$ (or $x < \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} z$ and $\mu_1 < \mu_2$), $x < \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} z$ and $\mu_1 > \mu_2$ (or $x > \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} z$ and $\mu_1 < \mu_2$), and $\mu_1 = \mu_2$, respectively. The mean–variance diagram will look like the curved shape in Fig. 1.

It is well known that either node 1 or node 2 is the risk-neutral 1-median that minimizes function $u(x)$. The following conclusions can be easily made from the figure for the risk-averse 1-median that maximizes $U(Y(x)) = -u(x) - \lambda v(x)$ when $\lambda > 0$ and the risk-seeking 1-median that maximizes $U(Y(x)) = -u(x) - \lambda v(x)$ when $\lambda < 0$:

- The risk-neutral 1-median will coincide with the risk-averse 1-median if it also minimizes function $v(x)$.
- The risk-neutral 1-median will coincide with the risk-seeking 1-median if it also maximizes function $v(x)$.

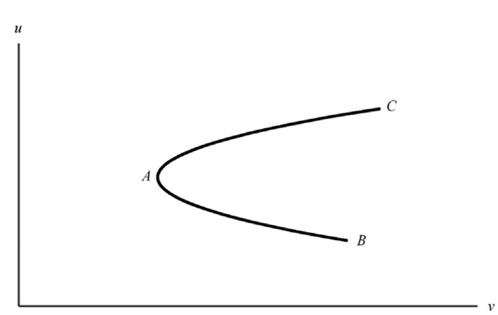

Fig. 1. Example 1: Mean–variance diagram.

- If the risk-neutral 1-median does not minimize function $v(x)$, then the risk-averse 1-median corresponds to a point somewhere between A and B on the curve in Fig. 1. As $u(x)$ and $v(x)$ are negatively correlated along the curve between A and B, the risk-averse solution reduces variability (i.e., the likelihood of long total weighted distances under unfavorable random demand scenarios) at the expense of increasing the mean total weighted distance.
- By Eq. (2), function $v(x)$ is convex over the segment. It follows that $U(Y(x))$ is a convex function of $x \in (0,z)$ if $\lambda < 0$, which implies that either node 1 or node 2 is the risk-seeking solution. Note that B and C in Fig. 1 correspond to these two nodal points. It is easy to see that the mean–variance combination at B (that is the risk-neutral 1-median) will also be the risk-seeking solution when the magnitude of λ is sufficiently small even if the risk-neutral 1-median is not a maximizer to function $v(x)$. If the mean–variance combination at C in Fig. 1 is the risk-seeking solution, however, it has a larger mean total weighted distance than the risk-neutral solution. Thus a risk-seeking 1-median may not be a sensible location if the total weighted distance to it is too long on average even though it gives the decision maker a chance to realize short total weighted distances under favorable random demand scenarios.

As an illustration, consider two scenarios of discrete probability distributions of the random weights presented in Table 2.

Let $z = 1$. For scenario (a), it is easy to verify $\mu_1 = 10$, $\mu_2 = 8.8$, $\sigma_1^2 = 1$ and $\sigma_2^2 = 15.36$. Note $u(x) = 10x + 8.8(1-x) = 8.8 + 1.2x$, and $v(x) = 1x^2 + 15.36(1-x)^2 = 16.36x^2 - 30.72x + 15.36$ at any point x on the segment. Given x, the objective function $U(Y(x))$ can be written as $U(Y(x)) = -u(x) - \lambda v(x) = -16.36\lambda x^2 - (1.2 - 30.72\lambda)x - 15.36\lambda - 8.8$.

The risk-averse solution under $\lambda = 0.5$ and the risk-neutral solution are $x = 0.87$ and $x = 0$, respectively. As shown in Table 3, the risk-averse solution appears better to a risk-averse decision maker than the risk-neutral solution because it ensures a more certain value of $-Y$, i.e., 8.35, 9.39, 10.09 and 11.13 (corresponding to demand weight realizations (9, 4), (9, 12), (11, 4) and (11, 12)) are close to its mean value of 9.84, and avoids a highly likely long total weighted distance of 12. This scenario suggests that a mean–variance objective is necessary to account for the risk of big losses.

For scenario (b), we have $\mu_1 = 8$, $\mu_2 = 8.4$, $\sigma_1^2 = 1$ and $\sigma_2^2 = 29.04$. It follows that $u(x) = 8x + 8.4(1-x) = 8.4 - 0.4x$, $v(x) = 1x^2 + 29.04(1-x)^2 = 30.04x^2 - 58.08x + 29.04$ and $U(Y(x)) = -u(x) - \lambda v(x) = -30.04\lambda x^2 + (0.4 + 58.08\lambda)x - 29.04\lambda - 8.4$.

The risk-neutral solution is $x = 1.0$. The risk-averse 1-median with $\lambda = 0.5$ and the risk-seeking 1-median with $\lambda = -0.5$ are $x = 0.98$ and $x = 0$, respectively. Since the risk-averse solution is very close to the risk-neutral solution, the distributions of $-Y$ for the risk-neutral solution and the risk-seeking solution only are presented in Table 3. Comparing the two distributions, we note that a decision maker may have reasons to be risk-seeking in this scenario. The mean total weight distance is 8.0 at the risk-neural 1-median, and 8.4 at the risk-seeking 1-median. Without increasing the weighted distance on average significantly, the risk-seeking 1-median could secure a much shorter weighted distance than the risk-averse 1-median or the risk-neutral 1-median if the realized demand at node 2 were 4, which would occur with a

Table 2
Example 1: Demand weight distributions.

(a)	h_1		h_2		(b)	h_1		h_2	
Probability	0.5	0.5	0.4	0.6	Probability	0.5	0.5	0.6	0.4
Demand	9	11	4	12	Demand	7	9	4	15

Table 3
Example 1: Probability distributions of $-Y$.

(a)	Risk-averse 1-median				Risk-neutral 1-median	
Probability	0.2	0.3	0.2	0.3	0.4	0.6
Weighted distance	8.35	9.39	10.09	11.13	4	12
(b)	Risk-seeking 1-median				Risk-neutral 1-median	
Probability	0.6		0.4		0.5	0.5
Weighted distance	4		15		7	9

probability of 0.6. However, the risk-seeking 1-median might also result in a weighted distance of 15 with a probability as high as 0.4.

It can be inferred from the analysis above that the probability distribution of the total weighted distance at the risk-neutral 1-median can help us predict the solution quality under different risk preferences. A risk-averse solution might be beneficial if the distribution has a either a high probability of unfavorable values or a likelihood of unacceptably large values, while a risk-seeking solution might be advantageous if the distribution has a high probability of favorable values (and therefore a low probability of undesirable values).

We note that under the mean–variance objective the closest assignment property (i.e., demand originating from each node is served by the closest open facility) does not necessarily hold for a multiple-facility location problem when the decision maker is not risk neutral, which will be evidenced by illustrative examples in Section 5. In order to avoid counter-intuitive solutions (e.g. customers residing at a node with an open facility are assigned to a facility somewhere else), we will discuss the closest assignment property for either model and enforce the closest assignment restriction when necessary. We further note that the loss of the closest assignment property is not unusual in facility location problems. Gerrard and Church [13] claimed that most network-based location models do not have this property and require some form of a closest assignment constraint to enforce it. In Section 5, we will use a numerical example to show that the closest assignment property is not necessarily guaranteed in a model to minimize CVaR.

3. The median problem

In this section we study the p-median problem with uncertain demand (p-MPUD). If the demand weight is deterministic at any node, the objective of the p-median problem is to locate p facilities so as to minimize the total weighted distance between the demand nodes and the facilities. Mathematically, the problem is to find a set of p points $\mathbf{X}_p$ such that $\sum_{i=1}^{n} h_i d(i,\mathbf{X}_p)$ is minimized. As a natural extension of this model to the domain of stochastic weights, p-MPUD is to maximize $U(Y)$ with $Y=-\sum_{i=1}^{n} h_i d(i,\mathbf{X}_p)$.

Let $F(\mathbf{X}_p)=E(-Y)+\lambda Var(-Y)$. Note

$$F(\mathbf{X}_p)=E\left[\sum_{i=1}^{n} h_i d(i,\mathbf{X}_p)\right]+\lambda Var\left[\sum_{i=1}^{n} h_i d(i,\mathbf{X}_p)\right]$$
$$=\sum_{i=1}^{n}\mu_i d(i,\mathbf{X}_p)+\lambda\sum_{i=1}^{n}\sum_{k=1}^{n} d(i,\mathbf{X}_p)d(k,\mathbf{X}_p)\sigma_{ik}, \quad (3)$$

with $\sigma_{ik}=\rho_{ik}\sigma_i\sigma_k$ being the covariance of random weights h_i and h_k. Since $U(Y)=-F(\mathbf{X}_p)$, p-MPUD is equivalent to

$$\min_{\mathbf{X}_p\subset G} F(\mathbf{X}_p). \quad (4)$$

If $\mathbf{X}_p$ is restricted to be a set of p nodes, then we call Eq. (4) the vertex p-MPUD. Otherwise, we have the absolute p-MPUD. It is easy to see that p-MPUD with $\lambda=0$ reduces to the p-median problem for which at least one set of p nodes is optimal, i.e., an optimal solution to the vertex p-MPUD is also optimal to the absolute p-MPUD. But vertex optimality in general does not carry over to $\lambda\neq 0$.

We next analyze $p=1$ and $p>1$, respectively, for $\lambda\neq 0$.

3.1. 1-MPUD

It is trivial to solve the vertex 1-MPUD. We now consider solving the absolute 1-MPUD. Arbitrarily select a link $(a,b)\in L$. Define an antipode y_i on link (a,b) as a point such that the distance from y_i to node i through node a is equal to the distance from y_i to node i through node b. Following Berman et al. [1], we refer to the region between two consecutive antipodes on link (a,b) as a *primary region* . Note that for any point inside a primary region on link (a,b) the sets of nodes optimally reachable via nodes a and b, respectively, remain unchanged. Denote these two sets by A and B, respectively.

For a point x inside a primary region on link (a,b), Eq. (3) reduces to

$$F(x)=\sum_{i\in A}\mu_i(x+d(i,a))+\sum_{i\in B}\mu_i(l_{ab}-x+d(i,b))$$
$$+\lambda\sum_{i\in A}\sum_{k\in A}(x+d(i,a))(x+d(k,a))\sigma_{ik}+2\lambda\sum_{i\in A}\sum_{k\in B}(x+d(i,a))(l_{ab}$$
$$-x+d(k,b))\sigma_{ik}+\lambda\sum_{i\in B}\sum_{k\in B}(l_{ab}-x+d(i,b))(l_{ab}-x+d(k,b))\sigma_{ik}.$$

The above expression can be rewritten in a quadratic function form of $F(x)=C_1x^2+C_2x+C_3$, where

$$C_1=\lambda\left(\sum_{i\in A}\sum_{k\in A}\sigma_{ik}-2\sum_{i\in A}\sum_{k\in B}\sigma_{ik}+\sum_{i\in B}\sum_{k\in B}\sigma_{ik}\right),$$

$$C_2=\sum_{i\in A}\mu_i-\sum_{i\in B}\mu_i+2\lambda\left\{\sum_{i\in A}\sum_{k\in A}d(i,a)\sigma_{ik}+\sum_{i\in A}\sum_{k\in B}[l_{ab}+d(k,b)\right.$$
$$\left.-d(i,a)]\sigma_{ik}-\sum_{i\in B}\sum_{k\in B}[l_{ab}+d(i,b)]\sigma_{ik}\right\},$$

$$C_3=\sum_{i\in A}\mu_i d(i,a)+\sum_{i\in B}[l_{ab}+d(i,b)]\mu_i+\lambda\left\{\sum_{i\in A}\sum_{k\in A}d(i,a)d(k,a)\sigma_{ik}\right.$$
$$+2\sum_{i\in A}\sum_{k\in B}d(i,a)[l_{ab}+d(k,b)]\sigma_{ik}+\sum_{i\in B}\sum_{k\in B}[l_{ab}+d(i,b)][l_{ab}$$
$$\left.+d(k,b)]\sigma_{ik}\right\}.$$

In the following lemma, we prove a structural property for function $F(x)$, which will facilitate finding an optimal solution.

Lemma 1. *Function $F(x)$ is (i) convex with respect to x within any primary region if $\lambda>0$ and (ii) concave if $\lambda<0$.*

Proof. Notice $C_1=\lambda Var(\sum_{i\in A}h_i-\sum_{i\in B}h_i)$. The convexity and concavity are thus established under $\lambda>0$ and $\lambda<0$, respectively. □

Based on the above lemma it is trivial to prove the theorem below.

Theorem 1. *Let x^* be an optimal solution of the 1-MPUD on a primary region $[r,s]$. (i) When $\lambda>0$, $x^*=-\frac{C_2}{2C_1}$ if $r<-\frac{C_2}{2C_1}<s$, and $x^*=\arg\min\{F(r),F(s)\}$ otherwise. (ii) When $\lambda<0$, $x^*=\arg\min\{F(r),F(s)\}$.*

Applying Theorem 1, we can find the optimal location inside every primary region and then the best one is chosen as the optimal solution to the entire network.

3.2. p-MPUD ($p>1$)

3.2.1. Absolute p-MPUD

The shortest paths to some nodes may shift as potential facility locations move along the links. In order to formulate the objective function explicitly in terms of the single facility location, we analyzed 1-MPUD in the previous sub-section over each primary region, in which such shifts cannot occur. If multiple facilities are to be located, determining the closest facility to each node may pose additional difficulty for deriving the objective function, which requires subdividing primary regions.

We now examine a set of points $\mathbf{X}_p=(x_{(1)},x_{(2)},\ldots,x_{(p)})$ within p selected primary regions, where $x_{(j)}$ denotes a point inside primary region $[r_{(j)},s_{(j)}]$ on link $(a_{(j)},b_{(j)})$ that is at a distance of $x_{(j)}$ away from node $a_{(j)}$. It is possible that two points $x_{(j)}$ and $x_{(m)}$ are located within the same primary region. For any node i, $d(i,x_{(j)})$ is equal to $d(i,a_{(j)})+x_{(j)}$ if $d(i,a_{(j)})+s_{(j)}\le d(i,b_{(j)})+l_{a_{(j)}b_{(j)}}-s_{(j)}$, or $d(i,b_{(j)})+l_{a_{(j)}b_{(j)}}-x_{(j)}$ otherwise. That is, the functional form of $d(i,x_{(j)})$ in terms of $x_{(j)}$ is invariant for $x_{(j)}\in[r_{(j)},s_{(j)}]$. Let δ be the set of constraints $r_{(j)}\le x_{(j)}\le s_{(j)}$.

Recall $d(i,\mathbf{X}_p)=\min 1\le j\le p\{d(i,x_{(j)})\}$ for node i. Given node i, we denote by $\varpi(i,x_{(j)})$ the set of constraints to ensure $d(i,x_{(j)})=d(i,\mathbf{X}_p)$. It is easy to see that this set contains linear constraints with respect to x_m $m=1,2,\ldots,p$. We call $\varpi(i,x_{(j)})$ feasible if there exists at least one set $\mathbf{X}_p$ with $r_{(m)}\le x_{(m)}\le s_{(m)}$ satisfying all the constraints.

Let Ω_i be the collection of feasible sets $\varpi(i,x_{(j)})$ $j=1,2,\ldots,p$ and Γ be the Cartesian product of Ω_i $i\in N$. A feasible element γ in Γ, which does not contain contradictory constraints with δ defines the functional form of $d(i,\mathbf{X}_p)$ for every node i and hence the objective function $F(\mathbf{X}_p)$ can be formulated in terms of points $(x_{(1)},x_{(2)},\ldots,x_{(p)})$. Suppose $d(i,\mathbf{X}_p)=d(i,x_{(j)})$. It follows that $d(i,\mathbf{X}_p)$ is a linear function of $x_{(j)}$. The conclusion below is natural.

Lemma 2. *Function $F(\mathbf{X}_p)$ with respect to $\mathbf{X}_p$ in the feasible region defined by δ and $\gamma\in\Gamma$ is quadratic and (i) convex if $\lambda>0$, and (ii) concave if $\lambda<0$.*

Proof. Note $F(\mathbf{X}_p)=f(d(1,\mathbf{X}_p),d(2,\mathbf{X}_p),\ldots,d(n,\mathbf{X}_p))$, where $f(q_1,q_2,\ldots,q_n)=\sum_{i=1}^n\mu_iq_i+\lambda\sum_{i=1}^n\sum_{k=1}^n q_iq_k\sigma_{ik}$. It is easy to see that the quadratic function $f(\cdot)$ is convex if $\lambda>0$ and concave if $\lambda<0$. The lemma follows for $F(\mathbf{X}_p)$ because $d(i,\mathbf{X}_p)$ is a linear function of $\mathbf{X}_p$ in the feasible region defined by δ and $\gamma\in\Gamma$. □

By the lemma, minimizing the objective function $F(\mathbf{X}_p)$ under constraint sets δ and $\gamma\in\Gamma$ is equivalent to minimizing a convex function when $\lambda>0$ or a concave function when $\lambda<0$ subject to linear constraints. The former can be solved as a convex quadratic programming model using the simplex method [36,31] and a global optimum is guaranteed. The latter is generally not a convex programming problem. Consequently, its local optima could differ from the global one. A global optimization algorithm such as the one proposed by Kalantari and Rosen [21] can be applied with an attempt to "escape" from a local optimum and move towards a global one.

The following procedure is designed to solve the absolute p-MPUD on a network:

- *Step* 1: Find the antipodes for every node on each link. Construct all primary regions.
- *Step* 2: For each possible combination of p primary regions (primary regions may be repetitive), implement Steps 3 to 5 and then go to Step 6 if all combinations have been fathomed.
- *Step* 3: Construct constraint sets δ and Γ.
- *Step* 4: For each feasible element $\gamma\in\Gamma$, find the optimal solution to minimize the objective function $F(\mathbf{X}_p)$ subject to constraints in δ and γ.
- *Step* 5: Compare all the optima obtained in Step 4 and choose the best one as the optimal solution for this combination of p primary regions. Return to Step 2.

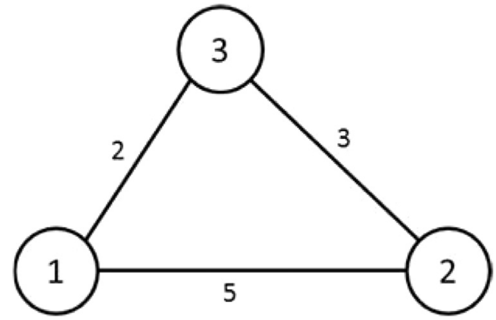

Fig. 2. Example 2: A three-node network.

- *Step* 6: Compare the optima for all possible combinations and choose the best one as the optimal solution for the entire network.

Example 2. To illustrate the exposition above, consider solving a 2-MPUD on the network presented in Fig. 2. The number next to each link in the figure reveals its length, while the parameters of random weights are $\mu_1=4$, $\sigma_1=2$, $\mu_2=3$, $\sigma_2=2$, $\mu_3=5$, $\sigma_3=3$, $\rho_{12}=-0.6$, $\rho_{13}=-0.5$, and $\rho_{23}=0.3$.

It is easy to verify that x=3.0 on link $(1,2)$ is an antipode of node 3, and that the intervals $[0,3]$ and $[3,5]$ are the two primary regions on link $(1,2)$, while the other two links are also primary regions, respectively.

In the Appendix, we present the process to solve the 2-median problem with λ=1 and -1 on two primary regions: the interval $[3,5]$ on link $(1,2)$ and link $(1,3)$.

It is noted that the algorithm presented above for the absolute p-MPUD is cumbersome even when the network size is small. Next, we consider the vertex p-MPUD for which exact solution algorithms are efficient at least for networks that are not too large.

3.2.2. Vertex p-MPUD

The vertex p-MPUD can be formulated as a mathematical programming model. We define the following binary variables:

$$Y_j=\begin{cases}1 & \text{if a facility is located at node } j,\\ 0 & \text{otherwise,}\end{cases}$$

$$X_{ij}=\begin{cases}1 & \text{if node } i \text{ is assigned to the facility located at node } j,\\ 0 & \text{otherwise.}\end{cases}$$

The vertex p-MPUD, referred to as (P_M), can now be formulated as follows:

$$\min\quad Z=\sum_{j=1}^n\sum_{i=1}^n\mu_id(i,j)X_{ij}+\lambda\sum_{j=1}^n\sum_{i=1}^n\sum_{l=1}^n\sum_{k=1}^n d(i,j)d(k,l)\sigma_{ik}X_{ij}X_{kl}$$

s.t.

$$\sum_{j=1}^n X_{ij}=1,\quad \text{for } i=1,2,\ldots,n$$

$$\sum_{j=1}^n Y_j=p$$

$$Y_j-X_{ij}\ge 0\quad \text{for } i,j=1,2,\ldots,n$$

$$\sum_{j=1}^n X_{ij}d(i,j)+(M-d(i,k))Y_k\le M\quad \text{for } i,k=1,2,\ldots,n$$

$$Y_j\in\{0,1\},\quad X_{ij}\in\{0,1\},\quad i,j=1,2,\ldots,n. \tag{5}$$

The constraint set (5) requires that customers be assigned to the closest facility (see [2]), where M is a very large number. One can define $M=\max_{i,j\in N}\{d(i,j)\}$. It is easy to see that a shorter distance between a demand node and the assigned facility is preferred in the above model when (i) the decision maker is risk neutral, or (ii) the decision maker is risk averse but the random demand weights are independent. Therefore, the closest assignment property is self-enforced and the constraint set (5) can be removed from the model in the above cases.

However, customers traveling to the closest facility is not certainly favorable to the mean–variance objective when $\lambda<0$ or when random demand weights are correlated even if $\lambda>0$. For instance, suppose that the weight associated with a demand node is negatively correlated with some others. Routing customers originating from that node to a facility far away may help reduce the variance of Y and thus improve the objective function when $\lambda>0$. As a result, the closest assignment property cannot be automatically guaranteed and thus the constraint set (5) has to be explicitly incorporated.

The above formulation is an integer quadratic programming model. Kariv and Hakimi [22] proved that the deterministic vertex p-median problem is NP-hard. Therefore, the vertex p-MPUD problem is also NP-Hard. Two exact solution approaches and a heuristic are developed below.

A branch and bound algorithm: Jarvinen et al. [19] presented a branch and bound algorithm to seek the deterministic p-median of a network. Extending its principle we construct a branch and bound algorithm for model (P_M). In the algorithm, the original problem is partitioned into many sub-problems, each of which divides the set of nodes N into two sub-sets: N^r contains the nodes where no facility is open $(1\le r\le n-p)$, while $N^{n-r}=N\backslash N^r$ contains the nodes that are potential sites to locate facilities. Note that a sub-problem yields a feasible location decision to the original problem when $r=n-p$.

We first generate sub-problems where only one node is closed. Subsequently, for each sub-problem with r closed nodes $(1\le r<n-p)$, we partition it by choosing one more node to be taken away and evaluate its lower bound. Given N^r and N^{n-r}, suppose that $\mathbf{X}_p\subset N^{n-r}$ is a set of locations to site p facilities. It is easy to verify that the objective function value can be written as

$$\begin{aligned}Z&=\sum_{i=1}^{n}\left\{\frac{1}{n}\mu_i d(i,\mathbf{X}_p)+\lambda[d(i,\mathbf{X}_p)]^2\sigma_i^2\right\}\\&\quad+\sum_{i=1}^{n}\sum_{k=1,k\ne i}^{n}\left\{\frac{1}{2n}[\mu_i d(i,\mathbf{X}_p)+\mu_k d(k,\mathbf{X}_p)]+\lambda d(i,\mathbf{X}_p)d(k,\mathbf{X}_p)\sigma_{ik}\right\}\\&=\sum_{i\in N^r}\sum_{k\in N^r}\tilde{R}_{ik}+2\sum_{i\in N^r}\sum_{k\in N^{n-r}}\tilde{R}_{ik}+\sum_{i\in N^{n-r}}\sum_{k\in N^{n-r}}\tilde{R}_{ik},\end{aligned}$$

where

$$\tilde{R}_{ik}=\begin{cases}\dfrac{1}{n}\mu_i d(i,\mathbf{X}_p)+\lambda[d(i,\mathbf{X}_p)]^2\sigma_i^2, & \text{if } i=k,\\ \dfrac{1}{2n}[\mu_i d(i,\mathbf{X}_p)+\mu_k d(k,\mathbf{X}_p)]+\lambda d(i,\mathbf{X}_p)d(k,\mathbf{X}_p)\sigma_{ik}, & \text{if } i\ne k.\end{cases}$$

Since the vector of the shortest distances $(d(1,\mathbf{X}_p),d(2,\mathbf{X}_p),\ldots,\mathbf{d}(\mathbf{n},\mathbf{X}_p))$ contains p zero terms, there exist at least p values of $k\in N^{n-r}$ such that $\tilde{R}_{ik}=\frac{\mu_i d(i,\mathbf{X}_p)}{2n}$ holds for every node $i\in N^r$ or $i\in N^{n-r}\backslash\mathbf{X}_p$. In addition, we have $\tilde{R}_{ii}=0$ and $\tilde{R}_{ik}=0$ for nodes $i,k\in\mathbf{X}_p$. It follows that $Z=\tilde{R}_1+2\tilde{R}_2+2\tilde{R}_3+\tilde{R}_4+2\tilde{R}_5$, where $\tilde{R}_1=\sum_{i\in N^r}\sum_{k\in N^r}\tilde{R}_{ik}$, $\tilde{R}_2=\sum_{i\in N^r}\sum_{k\in N^{n-r}\backslash\mathbf{X}_p}\tilde{R}_{ik}$, $\tilde{R}_3=\frac{p}{2n}\sum_{i\in N^r}\mu_i d(i,\mathbf{X}_p)$, $\tilde{R}_4=\sum_{i\in N^{n-r}\backslash\mathbf{X}_p}\sum_{k\in N^{n-r}\backslash\mathbf{X}_p}\tilde{R}_{ik}$, and $\tilde{R}_5=\frac{p}{2n}\sum_{i\in N^{n-r}\backslash\mathbf{X}_p}\mu_i d(i,\mathbf{X}_p)$.

Note that

$$\sum_{i=1}^{n}\min_{j\in N^{n-r}}\left\{\frac{1}{n}\mu_i d(i,j)+\lambda[d(i,j)]^2\sigma_i^2\right\}$$
$$+\sum_{i=1}^{n}\sum_{k=1,k\ne i}^{n}\min_{j\in N^{n-r}}\min_{l\in N^{n-r}}\left\{\frac{1}{2n}[\mu_i d(i,j)+\mu_k d(k,l)]+\lambda d(i,j)d(k,l)\sigma_{ik}\right\} \quad (6)$$

bounds the objective function of a sub-problem (N^r, N^{n-r}) from below. Let

$$R_{ik}=\begin{cases}\min_{j\in N^{n-r},j\ne i}\{\frac{1}{n}\mu_i d(i,j)+\lambda[d(i,j)]^2\sigma_i^2\}, & \text{if } i=k,\\ \min_{j\in N^{n-r},j\ne i}\min_{l\in N^{n-r},l\ne k}\{\frac{1}{2n}[\mu_i d(i,j)+\mu_k d(k,l)]+\lambda d(i,j)d(k,l)\sigma_{ik}\}, & \text{if } i\ne k\end{cases}$$

for each pair of nodes (i, k) and

$$R_i'=\frac{\mu_i}{2n}\min_{j\in N^{n-r},j\ne i}d(i,j)$$

for each node i. It can be shown that $LB=R_1+2R_2+2R_3+R_4+2R_5$ is a lower bound of the sub-problem with $R_1=\sum_{i\in N^r}\sum_{k\in N^r}R_{ik}$, $R_2=\sum_{i\in N^r}$ the sum of the $n-r-p$ smallest values of R_{ik} $(k\in N^{n-r})$, $R_3=p\sum_{i\in N^r}R_i'$, $R_4=$ the sum of the $n-r-p$ smallest values of R_{ii} $(i\in N^{n-r})$ $+2\times$ the sum of the $\frac{(n-r-p)(n-r-p-1)}{2}$ smallest values of R_{ik} $(i,k\in N^{n-r}, i\ne k)$, and $R_5=p\times$ the sum of the $n-r-p$ smallest values of R_i' $(i\in N^{n-r})$.

The branch and bound algorithm is presented below:

- *Step* 1: $r=1, U=\infty$. Generate sub-problems $N^r=\{1\},\{2\},\ldots,\{n\}$. Go to Step 3.
- *Step* 2: Let r be the number of closed nodes associated with sub-problem P'. Increase r by 1, delete sub-problem P', and generate sub-problems for which all nodes taken away in P' and an additional node are closed. To avoid repetition, the new node to be closed should have an index higher than the existing ones.
- *Step* 3: Compute LB for each newly generated sub-problem. Denote the sub-problem with the smallest LB by P'.
- *Step* 4: If $r<n-p$, then go to Step 2.
- *Step* 5: If $r=n-p$, then P' is a feasible solution of model (P_M). Update U with LB of P'. Delete any sub-problem with $LB\ge U$. If there exists no sub-problem, then the algorithm terminates; otherwise, denote the sub-problem with the smallest LB by P' and go to Step 2.

Linearization: Glover's [14] linearization method is applied in our study to solve the vertex p-MPUD as follows.

Define

$$W_{ij}=d(i,j)X_{ij}\sum_{l=1}^{n}\sum_{k=1}^{n}d(k,l)\sigma_{ik}X_{kl},$$

$$C_{ij}^{+}=d(i,j)\sum_{l=1}^{n}\sum_{k=1}^{n}d(k,l)\sigma_{ik}1_{\{\rho_{ik}>0\}},$$

$$C_{ij}^{-}=d(i,j)\sum_{l=1}^{n}\sum_{k=1}^{n}d(k,l)\sigma_{ik}1_{\{\rho_{ik}<0\}}.$$

We can replace $d(i,j)X_{ij}\sum_{l=1}^{n}\sum_{k=1}^{n}d(k,l)\sigma_{ik}X_{kl}$ in the objective function of model (P_M) with W_{ij}. Notice that in the formulation below, W_{ij} is equal to $d(i,j)\sum_{l=1}^{n}\sum_{k=1}^{n}d(k,j)\sigma_{ik}X_{kl}$ when X_{ij} is 1, and W_{ij} is equal to 0 when X_{ij} is 0. Model (P_M) can be written as a mixed-integer linear programming model:

$$\min \quad Z=\sum_{j=1}^{n}\sum_{i=1}^{n}[\mu_i d(i,j)+\lambda W_{ij}]$$

s.t.

constraints of (P_M) and

$$C_{ij}^{-}X_{ij}\le W_{ij}\le C_{ij}^{+}X_{ij}, \quad \text{for } i,j=1,2,\ldots,n$$

$$W_{ij}\ge d(i,j)\sum_{l=1}^{n}\sum_{k=1}^{n}d(k,l)\sigma_{ik}X_{kl}-C_{ij}^{+}(1-X_{ij}), \quad \text{for } i,j=1,2,\ldots,n$$

$$W_{ij} \le d(i,j)\sum_{l=1}^{n}\sum_{k=1}^{n} d(k,l)\sigma_{ik}X_{kl} - C_{ij}^{-}(1-X_{ij}), \quad \text{for } i,j=1,2,\ldots,n$$

$$W_{ij} \in \mathbb{R}, i,j=1,2,\ldots,n. \tag{7}$$

The theorem below indicates that the above model returns the optimal solution to p-MPUD.

Theorem 2. *The vertex p-MPUD is equivalent to model (7).*

Proof. The theorem is straightforward because the corresponding feasible solutions (Y_j, X_{ij}) to models (P_M) and (Y_j, X_{ij}, W_{ij}) to model (7) have an identical objective function value. □

An interchange procedure: This heuristic first finds a feasible solution using a greedy algorithm and then attempts to search for better solutions by swapping a node in the current solution with a node that does not have an open facility. The pair of nodes for interchange is such that the objective function is improved as much as possible. This process continues until no improvement could be made at the current solution. The procedure is formally stated below:

- *Step* 1: Let $\mathbf{X} = \varnothing$, $r=0$, $UB = \infty$.
- *Step* 2: For any node $j \in N\backslash\mathbf{X}$, if $F(\mathbf{X} \cup \{j\}) < UB$, then let $UB = F(\mathbf{X} \cup \{j\})$ and $j^* = j$.
- *Step* 3: Let $r = r+1$ and $\mathbf{X} = \mathbf{X} \cup \{j^*\}$. If $r < p$, then let $UB = \infty$ and go to Step 2.
- *Step* 4: For each pair of nodes (j,k) $j \in N\backslash\mathbf{X}$ and $k \in \mathbf{X}$, if $F(\mathbf{X} \cup \{j\}\backslash\{k\}) < UB$, then let $UB = F(\mathbf{X} \cup \{j\}\backslash\{k\})$ and $(j^*, k^*) = (j,k)$.
- *Step* 5: If UB was not improved in Step 4, then return $\mathbf{X}$. Otherwise, let $\mathbf{X} = \mathbf{X} \cup \{j^*\}\backslash\{k^*\}$ and go to Step 4.

4. The center problem

The deterministic p-center problem is to seek locations for p facilities with the objective of minimizing the maximum weighted distance from a demand node to the closest facility. Under the mean–variance objective, the center problem with uncertain demand (p-CPUD) can be formulated as

$$\max_{\mathbf{X}_p \subset G} \min_{i \in N} U(-h_i d(i,\mathbf{X}_p)) = \max_{\mathbf{X}_p \subset G} \min_{i \in N}\big(E(-h_i d(i,\mathbf{X}_p)) - \lambda Var(-h_i d(i,\mathbf{X}_p))\big) = -\min_{\mathbf{X}_p \subset G} \max_{i \in N}[\mu_i d(i,\mathbf{X}_p) + \lambda(d(i,\mathbf{X}_p))^2\sigma_i^2]. \tag{8}$$

We note that correlation between random demand weights does not affect the optimal solution to the above model. In addition, the closest assignment property is automatically implied if $\lambda \ge 0$ as shorter distances are always preferred.

For a given node $i \in N$, let $F_i(\mathbf{X}_p) = \mu_i d(i,\mathbf{X}_p) + \lambda(d(i,\mathbf{X}_p))^2\sigma_i^2$. p-CPUD is equivalent to

$$\min_{\mathbf{X}_p \subset G} \max_{i \in N} F_i(\mathbf{X}_p). \tag{9}$$

4.1. 1-CPUD

We start with $p=1$. Arbitrarily select a primary region $[r,s]$ on link (a,b). Using the notations defined in Section 3, we have

$$d_i(x) = \begin{cases} d(i,a)+x, & \text{if } i \in A, \\ d(i,b)+l_{ab}-x, & \text{if } i \in B \end{cases}$$

for any point $x \in [r,s]$. Note that $F_i(x)$ is a convex quadratic function within a primary region when $\lambda > 0$. Since the maximum of convex functions is convex, $C(x) = \max_{i \in N} F_i(x)$ is a convex function and therefore has a unique minimum over every primary region. Note that the stationary point of function $F_i(x)$ is

$$x = \begin{cases} -\left(\frac{\mu_i}{2\lambda\sigma_i^2} + d(i,a)\right), & \text{if } i \in A, \\ \frac{\mu_i}{2\lambda\sigma_i^2} + l_{ab} + d(i,b), & \text{if } i \in B, \end{cases}$$

which is outside the link (a, b). Hence $F_i(x)$ over a primary region is a monotone increasing function if $i \in A$ or a monotone decreasing function if $i \in B$.

Kariv and Hakimi [23] constructed a set of dominant points for the deterministic 1-center problem on a link. We can make similar claims: (i) a dominant point on link (a, b) for 1-CPUD is $x=0$ (node a), or $x = l_{ab}$ (node b), or x is a point where two functions $F_i(\cdot)$ and $F_j(\cdot)$ intersect inside a primary region with $i \in A$ and $j \in B$. (ii) Let $\mathbf{X}$ be the set of these dominant points. The optimal solution to 1-CPUD on link (a, b) is $x^* = \arg\min_{x \in \mathbf{X}} C(x)$.

Therefore, to solve 1-CPUD on a link, we can simply find all the dominant points and then implement claim (ii) stated above. The monotonicity of function $F_i(\cdot)$ within each primary region implies that the antipode associated with node i, if available, is a maximum point of function $F_i(\cdot)$ over the link. We note that it cannot be a global minimum point of function $C(\cdot)$ unless it belongs to the dominant set $\mathbf{X}$.

Example 3. As an example, we revisit the network in Fig. 2. The mean and standard deviation of each demand node are as follows: $\mu_1 = 2, \mu_2 = 2, \mu_3 = 5, \sigma_1 = 2, \sigma_2 = 1.5$ and $\sigma_3 = 1$. We also assume $\lambda = 1$. Let us examine link $(1,2)$.

Recall that link $(1,2)$ consists of two primary regions $[0,3]$ and $[3,5]$. It is easy to derive $F_1(x) = 2x + 4x^2$, $F_2(x) = 2(5-x) + 2.25(5-x)^2$ and $F_3(x) = \begin{cases} 5(2+x)+(2+x)^2, & 0 \le x \le 3, \\ 5(8-x)+(8-x)^2, & 3 \le x \le 5. \end{cases}$

A plot of functions $F_i(x)$'s is given in Fig. 3.

The dominant set can be constructed as $\mathbf{X} = \{0, 1.663, 2.185, 3.192, 5.0\}$ because functions $F_2(\cdot)$ and $F_3(\cdot)$, $F_1(\cdot)$ and $F_2(\cdot)$, $F_1(\cdot)$ and $F_3(\cdot)$ meet at x=1.663, 2.185 and 3.192, respectively. The optimal solution on link (1, 2) is $x^* = 1.663$ with an objective function value $C(x^*) = 31.729$.

For any node i, $F_i(x)$ is a concave function (though not necessarily monotone increasing or decreasing) within a primary region when $\lambda < 0$. Based on Zangwill's [37] Theorems 1 and 4, the maximum of concave functions is a piecewise concave function. Similar to the case of $\lambda > 0$, we conclude (i) a dominant point on link (a, b) for 1-CPUD is $x=0$ (node a), or $x = l_{ab}$ (node b), or x is a point where two functions $F_i(\cdot)$ and $F_j(\cdot)$ intersect inside a primary region with derivatives of opposite signs, or x is an antipode

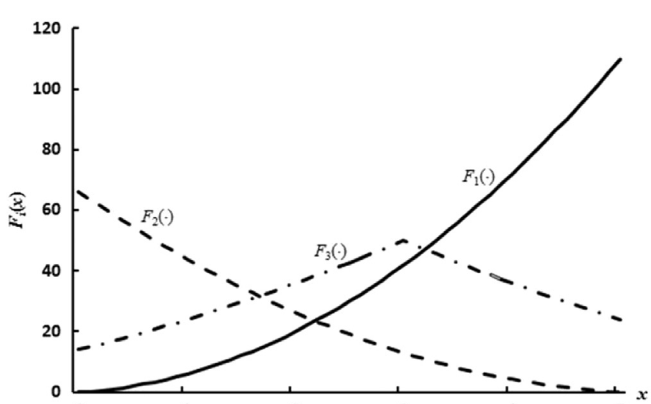

Fig. 3. Example 3: Functions $F_i(\cdot)$ when $\lambda = 1$.

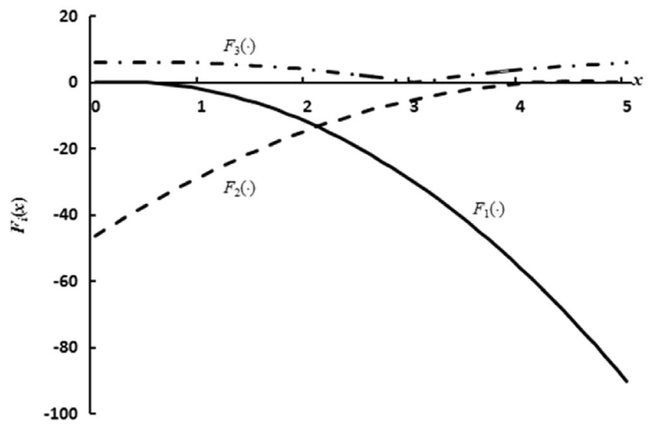

Fig. 4. Example 3: Functions $F_i(\cdot)$ when $\lambda = -1$.

associated with node i when $\frac{dF_i(y)}{dy}|_{y \to x^-} < 0$ and $\frac{dF_i(y)}{dy}|_{y \to x^+} > 0$. (ii) Let $\mathbf{X}$ be the set of these dominant points over link (a, b). The optimal solution to 1-CPUD on link (a, b) is $x^* = \arg\min_{x \in \mathbf{X}} C(x)$.

Note that in contrast to $\lambda > 0$, the antipode associated with node i might be the minimum point of function $F_i(\cdot)$ over the link. Such an antipode belongs to the dominant set.

We consider again the example above for $\lambda = -1$. It is easy to verify $F_1(x) = 2x - 4x^2$, $F_2(x) = 2(5-x) - 2.25(5-x)^2$ and $F_3(x) = \begin{cases} 5(2+x) - (2+x)^2, & 0 \le x \le 3, \\ 5(8-x) - (8-x)^2, & 3 \le x \le 5. \end{cases}$

A plot of functions $F_i(x)$'s is presented in Fig. 4.

In this example, the dominant set $\mathbf{X} = \{0, 2.088, 3.0, 5.0\}$ because functions $F_1(\cdot)$ and $F_2(\cdot)$ meet at x=2.088, and x=3.0 is the minimum point of function $F_3(\cdot)$ as well as the antipode associated with node 3. It is easy to verify that $x^* = 3.0$ is optimal on link (1,2).

4.2. p-CPUD ($p > 1$)

We subsequently extend our analysis to the multiple-facility case. By Theorem 3.1 in Garfinkel et al. [12], we can show that the dominant set for p-CPUD is identical to that for 1-CPUD. Without loss of generality let us assume that N is the acceptable dominant set. p-CPUD can be formulated as a quadratic mathematical programming model, which we call (P_C), using the same binary variables defined for model (P_M):

$$\min \quad Z \qquad\qquad (P_C)$$

s.t.

constraints of model (P_M) and

$$Z \ge \sum_{j=1}^{n} [\mu_i d(i,j) X_{ij} + \lambda (d(i,j))^2 \sigma_i^2 X_{ij}^2], \quad \text{for } i = 1, 2, \dots, n.$$

Note that since X_{ij} is binary, we can easily convert the above quadratic programming formulation to a mixed-integer linear programming formulation, where the last set of constraints changes to

$$Z \ge \sum_{j=1}^{n} [\mu_i d(i,j) + \lambda (d(i,j))^2 \sigma_i^2] X_{ij}, \quad \text{for } i = 1, 2, \dots, n.$$

As discussed earlier in this section, the closest assignment constraint set (5) is necessary only when $\lambda < 0$. In this study, we solve the above model using the mixed-integer linear programming package Xpress MP.

5. Numerical analysis and insights

We start this section with a sensitivity analysis of an absolute 1-MPUD to investigate the effects of risk attitudes, expected values, variances and correlations of the demand weights on the optimal location. The optimal solutions to a vertex 2-MPUD are then examined to discuss the closest assignment property and the impact of the number of facilities on the decision maker's utility in the location models with the mean–variance objective.

Optimal solutions have been obtained under various scenarios, e.g., independent and dependent random weights and different mean values and variances of the demand weights, as well as various number of facilities to locate. The observations are similar in nature under all these scenarios for the two problems.

Example 4. In order to easily explore properties and gain insights, we choose to focus on a line of ten nodes that are one unit away from the neighboring nodes. The nodes are labelled as 1–10 in sequence along the line.

5.1. Sensitivity analysis

In the *baseline*, each independent demand weight has a mean value of 20 and a variance of 36. In each scenario, we change the mean value or variance of a selected demand weight only. For the baseline and all scenarios, the absolute 1-MPUD is solved for $\lambda = \pm 0.01, \pm 0.05, \pm 0.1$ and ± 20.0, respectively. The computational results are available in Tables 4 and 5 for the baseline and selected scenarios in which the mean value or the variance of the demand weight at node 1 or node 3 takes a different value. Node 1 and node 3 are chosen so as to examine the relevance of a node's location relative to the others. In the tables, each optimal location is expressed by its distance to node 1.

The “min-var” median problem and the “max-var” median problem can be formulated as

$$\min_{\mathbf{X}_p \subset G} \sum_{i=1}^{n} \sum_{k=1}^{n} d(i, \mathbf{X}_p) d(k, \mathbf{X}_p) \sigma_{ik},$$

$$\max_{\mathbf{X}_p \subset G} \sum_{i=1}^{n} \sum_{k=1}^{n} d(i, \mathbf{X}_p) d(k, \mathbf{X}_p) \sigma_{ik}.$$

It is easy to see that they represent the extreme scenarios of risk averse and risk seeking, respectively. The optimal locations to the above two models when p=1 are referred to as the min-var median and the max-var median in Table 4.

We have the following findings and insights from the tables:

- The median may differ substantially with risk attitudes. In most of the instances presented, the optimal location tends to

Table 4
Example 4: Risk-neutral median, min-var median and max-var median.

Scenario	Risk-neutral median	Min-var median	Max-var median
Baseline	$x \in [4.0, 5.0]$	4.5	(0.0, 9.0)
$\mu_1 = 5$	5.0	4.5	(0.0, 9.0)
$\mu_1 = 10$	5.0	4.5	(0.0, 9.0)
$\mu_1 = 40$	4.0	4.5	(0.0, 9.0)
$\mu_1 = 80$	3.0	4.5	(0.0, 9.0)
$\mu_3 = 5$	5.0	4.5	(0.0, 9.0)
$\mu_3 = 80$	3.0	4.5	(0.0, 9.0)
$\sigma_1^2 = 9$	$x \in [4.0, 5.0]$	4.86	0.0
$\sigma_1^2 = 18$	$x \in [4.0, 5.0]$	4.74	0.0
$\sigma_1^2 = 72$	$x \in [4.0, 5.0]$	4.09	9.0
$\sigma_1^2 = 144$	$x \in [4.0, 5.0]$	3.46	9.0
$\sigma_3^2 = 9$	$x \in [4.0, 5.0]$	4.7	9.0
$\sigma_3^2 = 144$	$x \in [4.0, 5.0]$	3.92	9.0

Table 5
Example 4: Risk-averse median and risk-seeking median.

λ	0.01	0.05	0.5	20.0
Baseline	4.5	4.5	4.5	4.5
$\mu_1 = 5$	5.0	4.93	4.71	4.5
$\mu_1 = 10$	5.0	4.78	4.64	4.5
$\mu_1 = 40$	4.0	4.0	4.22	4.5
$\mu_1 = 80$	3.0	3.96	4.0	4.5
$\mu_3 = 5$	5.0	4.92	4.71	4.5
$\mu_3 = 80$	3.0	3.94	4.0	4.5
$\sigma_1^2 = 9$	4.86	4.86	4.86	4.86
$\sigma_1^2 = 18$	4.74	4.74	4.74	4.74
$\sigma_1^2 = 72$	4.09	4.09	4.09	4.09
$\sigma_1^2 = 144$	4.0	4.0	3.89	3.46
$\sigma_3^2 = 9$	4.7	4.7	4.7	4.7
$\sigma_3^2 = 144$	4.0	4.0	4.0	3.92
λ	−0.01	−0.05	−0.1	−20.0
Baseline	(4.0,5.0)	(4.0,5.0)	(0.0,9.0)	(0.0,9.0)
$\mu_1 = 5$	5.0	8.0	9.0	9.0
$\mu_1 = 10$	5.0	7.0	9.0	9.0
$\mu_1 = 40$	4.0	0.0	0.0	0.0
$\mu_1 = 80$	3.0	0.0	0.0	0.0
$\mu_3 = 5$	5.0	8.0	9.0	9.0
$\mu_3 = 80$	3.0	2.0	0.0	0.0
$\sigma_1^2 = 9$	4.0	3.0	0.0	0.0
$\sigma_1^2 = 18$	4.0	4.0	0.0	0.0
$\sigma_1^2 = 72$	5.0	9.0	9.0	9.0
$\sigma_1^2 = 144$	5.0	9.0	9.0	9.0
$\sigma_3^2 = 9$	4.0	3.0	0.0	0.0
$\sigma_3^2 = 144$	5.0	9.0	9.0	9.0

significantly deviate from the risk-neutral median when $|\lambda|$ is not so small. For instance, the risk-neutral median on the line is $x=4.0$ (i.e., node 5) when $\mu_1 = 40$, $\mu_i = 20$ for $i > 1$ and $\sigma_i^2 = 36$ for $1 \le i \le 10$. However, the risk-averse median is $x=4.5$ (the mid-point between node 5 and node 6) for $\lambda = 20.0$ and the risk-seeking median is $x=0.0$ (node 1) for $\lambda = -0.05$, -0.1, or -20.0. It is thus apparent that incorporating the risk attitude into location models has a considerable effect.

- The optimal location balances the mean and variance of variable Y. The risk-averse median and the risk-seeking median are optimal to the risk-neutral median problem when λ is close to zero. As the decision maker's risk attitude departs from neutrality, the effect of the variance of the demand weights becomes pronounced. Therefore, the risk-averse median and the risk-seeking median approach the min-var median and the max-var median (or one of the max-var medians), respectively, when the magnitude of λ is large enough.
- The risk-seeking median is optimal to the max-var median problem when λ is no larger than -0.05 or -0.1. However, in most of the instances solved the risk-averse median becomes identical to the min-var median only when λ is as large as 20. This contrast is due to the attraction of the risk-neutral median to the risk-averse median objective function as a shorter expected total distance is generally favorable to the minimization of the variance of Y. Consequently, the risk-averse median deviates from the risk-neutral median at a slower rate as λ increases. On the contrary, the max-var median is far away from the risk-neutral median, which suggests that the variance of Y at the risk-neutral median is not so small. Hence, the risk-seeking median is easily dragged away from the risk-neutral median as a decreasing negative λ would give a heavier weight to the variance of Y.
- Our sensitivity analysis results show that (i) the risk-averse median tends to move away from the node with a decreasing mean or variance of the demand weight and move towards the node with an increasing mean or variance of the demand weight; while (ii) when the random demand weights are independent, the risk-seeking median tends to move away from the node with a decreasing mean or an increasing variance of the demand weight and move towards the node with an increasing mean or a decreasing variance of the demand weight. However, the movement of the risk-seeking median is not very consistent when the demand weights are dependent as the trade-off between the mean and variance of Y is more subtle due to correlations.
- It is noted that the risk-averse median is more sensitive to changes in the variance of the demand weight at a farther node. On the contrary, when a mean demand weight associated with a node changes, the distance of the node to the original risk-averse median seems to have no significant impact. For instance, given $\lambda = 0.05$, the risk-averse median at the baseline is $x=4.5$, while we have $x=4.86$ when $\sigma_1^2 = 9$, $x=4.7$ when $\sigma_3^2 = 9$, $x=4.92$ when $\mu_1 = 5$, and $x=4.93$ when $\mu_3 = 5$.

The above findings except the one noted regarding the movement of the risk-seeking median also apply to instances with dependent random demand weights that have been solved.

5.2. Closest assignment

As argued in Section 2, the closest assignment restriction has to be explicitly enforced in the multiple-facility location models under some conditions. Take the vertex p-MPUD ($p > 1$) as an example. It was noted in Section 3 that the closest assignment constraint set (5) should be included in model (P_M) when (i) $\lambda < 0$ or (ii) $\lambda > 0$ and not all random demand weights are independent. We next show that the optimal solution to the vertex 2-MPUD in Example 4 would not make sense if the closest assignment constraint set was not incorporated.

The vertex 2-MPUD problem is solved under the following two scenarios: (a) demand weights are independent; (b) demand weights associated with node 1, node 9 and node 10 are correlated with $\rho_{1,9} = \rho_{1,10} = -0.8$ and $\rho_{9,10} = 0.1$. In both scenarios each independent demand weight has a mean value of 20 and a variance of 36.

Model (P_M) is solved with and without the closest assignment constraint set (5) for both scenarios with $\lambda = -1$ and $\lambda = 3.5$. Table 6 presents the optimal locations and the maximum utility for the instances solved.

It is easy to see from the table that the closest assignment constraint set (5) affects the optimal locations except for $\lambda=3.5$ under scenario (a). When the closest assignment restriction is not enforced, at least some of the demand nodes are assigned to the more distant facility in all the other instances. For instance, the vertex 2-median under scenario (b) is nodes 1 and 10 when

Table 6
Optimal solutions to model (P_M).

Scenario	(a)		(b)	
With the closest assignment constraint set				
λ	−1	3.5	−1	3.5
Optimal locations	(1, 2) or (9, 10)	(3, 8)	(1, 2)	(4, 7)
Maximum utility	6624.0	−2760.0	7027.2	−1187.2
Without the closest assignment constraint set				
λ	−1	3.5	−1	3.5
Optimal locations	(1, 10)	(3, 8)	(1, 10)	(3, 6)
Maximum utility	16,960.0	−2760.0	14,742.4	−914.4
Risk-neutral solution (3, 8)				
λ	−1	3.5	−1	3.5
Utility	480.0	−2760.0	148.8	−1600.8

Table 7
Example 5: Discrete probability functions of $-Y$.

Scenario	(a)			(b)			
Distance	10	50	100	130	140	170	210
Probability	0.5	0.3	0.2	0.3	0.2	0.3	0.2

$\lambda = -1$. However, it would not be optimal for customers residing on any node other than node 1 to travel to the closer facility. In particular, customers originating from node 10 with an open facility have to patron the facility located on node 1. This example clearly shows that longer distances may affect the variance in a way favorable to the decision maker's utility. Therefore, the closest assignment restriction is necessary in order to avoid counter-intuitive solutions.

Note that the closest assignment property always holds for the risk-neutral models. The risk-neutral solution to the above vertex 2-MPUD instances is to locate facilities at node 3 and node 8. This solution yields a utility of -240 when $\lambda = 0$. To compare the quality of solutions under different risk attitudes, the utility of this risk-neutral solution is computed for scenario (a) or scenario (b), with the risk attitude coefficient $\lambda = -1$ or $\lambda = 3.5$, and presented in Table 6. It is evident that the risk-neutral solution is much inferior to the risk-averse or the risk-seeking solution when risk is taken into account. We thus infer that it is beneficial to apply the risk-averse and risk-seeking models over the risk-neutral model even when the closest assignment constraint set is incorporated.

As shorter distances do not always improve the mean–variance objective function, increasing the number of facilities to open does not necessarily raise the utility of a decision maker who is not risk-neutral when customers are always served by the closest facility. It is easy to infer that this effect should be frequently observed when the decision maker is risk-seeking as longer distances tend to increase the variance of variable Y. However, the following numerical example suggests that it exists even when the decision maker is risk-averse. We solve the vertex p-MPUD on the line with $p = 1$ and $p = 2$, respectively. The demand weights associated with node 1, node 7, node 8 node 9 and node 10 are correlated with $\rho_{1,7} = \rho_{1,8} = \rho_{1,9} = \rho_{1,10} = -0.8$, $\rho_{7,10} = \rho_{8,10} = \rho_{9,10} = -0.1$, $\rho_{7,8} = 0.01$ and $\rho_{7,9} = \rho_{8,9} = 0.05$. Given $\lambda = 3.5$, the vertex 1-median is node 5 with a utility of 946.48, while the vertex 2-median is nodes 5 and 6 with a utility of 590.36 only.

In our study, closest assignment does not automatically occur even for a risk-averse model when some of the weights are negatively correlated because the risk measure (variance) could be reduced by longer distances. It is thus conceivable that other risk measures including downside-risk ones, may lead to the same problem.

Example 5. As an illustration, we apply minimizing CVaR of $-Y = \sum_{i=1}^{n} h_i d(i, \mathbf{X}_p)$ as the objective for a simple vertex 2-median problem on a line of three nodes that are ten units away from the neighboring nodes. The nodes are labelled as 1 to 3 in sequence along the line. The random demand weights for the three nodes have the following joint probability function: $P(h_1 = 2, h_2 = 10, h_3 = 3) = 0.1$, $P(h_1 = 2, h_2 = 10, h_3 = 6) = 0.1$, $P(h_1 = 6,\ h_2 = 5, h_3 = 6) = 0.3$, $P(h_1 = 6, h_2 = 1, h_3 = 6) = 0.3$, and $P(h_1 = 10,\ h_2 = 1, h_3 = 12) = 0.2$. Suppose that nodes 1 and 3 host the two facilities. Table 7 presents the probability distributions of the total weighted distance $-Y$ under scenario (a) the closest assignment rule, and scenario (b) routing both node 1 and node 2 to the open facility at node 3.

Let the confidence level be 0.8. It is easy to verify for $-Y$ we have VaR = 50 and CVaR = 50 under scenario (a), and VaR = 170 and CVaR = 40 under scenario (b). That is, closest assignment cannot be guaranteed under the objective of CVaR.

Table 8
Average CPU time (in seconds).

n	p	p-MPUD			p-CPUD
		LR	*BB*	*INT*	*LR*
10	2	4.8	0.1	0.1	0.3
10	4	3.3	0.1	0.1	0.1
10	6	1.8	0.1	0.1	0.1
25	5	276.8	107.0	0.1	0.8
25	10	3530.8	4184.0	0.1	1.8
25	15	1310.0	3124.3	0.1	0.6
50	10	10,800	10,800	0.1	4.3
50	20	10,800	10,800	0.1	3.5
50	30	10,800	10,800	0.1	1.3
75	15	10,800	10,800	0.1	16.8
75	30	10,800	10,800	0.1	12.0
75	45	10,800	10,800	0.1	5.3
100	20	n/a	10,800	0.8	100.3
100	40	n/a	10,800	1.3	34.8
100	60	n/a	10,800	2.5	31.0
150	30	n/a	10,800	15.5	118.8
150	60	n/a	10,800	19.0	116.5
150	90	n/a	10,800	11.5	98.8
200	40	n/a	10,800	46.0	560.5
200	80	n/a	10,800	20.3	303.8
200	120	n/a	10,800	22	497.8

Table 9
Hit rates and average optimality gaps (p-MPUD).

n	Hit rate (%)			Average gap (%)		
	LR	*BB*	*INT*	*LR*	*BB*	*INT*
10	100	100	100	0	0	0
25	100	100	58.3	0	0	5.7
50	50	100	100	27.1	0	0
75	50	50	50	67.2	35.0	35.0
100	n/a	100	100	n/a	0	0
150	n/a	100	100	n/a	0	0
200	n/a	100	100	n/a	0	0

6. Computational experiment

A computational experiment was carried out to test the performance of the linearization method as well as the other algorithms in terms of CPU time and solution quality. All the solution procedures were coded in Microsoft Visual C++ and run on a PC with Intel Core i5-1.70G Hz CPU and 8 GB RAM.

We used the following parameter values to construct test instances for each problem studied: Networks were randomly generated for the number of nodes $n = 10$, 25, 50, 75, 100, 150 and 200; n nodes were randomly positioned inside a square with a side length of 100 units; the distance matrix $\{d_{ij}\}$ was evaluated using the Euclidean distances; the mean and standard deviation of each demand weight and the correlation coefficient between a pair of demand weights were randomly chosen; the number of facilities $p = 0.2n$, $0.4n$ and $0.6n$; the risk attitude coefficient $\lambda = -5, -1, 1$ and 5. In total, 84 test instances were generated for $p-$ MPUD and $p-$ CPUD. In the experiment, we assumed that the nodes of a network were potential location sites.

The mixed-integer linear programming models obtained after linearization were solved by the MIP optimizer in Xpress-MP. However, because the number of decision variables and constraints in model (7) increases quickly with n, the optimizer either took very long time to solve instances or could not solve them at all due to insufficient memory when n was large enough. We hence also ran the branch and bound algorithm (*BB*) and the interchange procedure (*INT*) in the experiment to solve p-MPUD. A time limit of 10,800 s was imposed on both the linearization method (*LR*) and the branch and bound algorithm. The results are summarized in Tables 8 and 9.

It is easy to see from Table 8 that the CPU time spent by each procedure grew with n. It took the Xpress-MP MIP optimizer up to 1203 s to solve a p-CPUD instance with 200 nodes. The time needed for $\lambda > 0$ was much shorter because the closest assignment constraint set (5) was not included. Therefore, this exact solution method is satisfactory for p-CPUD.

It is not surprising that *INT* ran very fast to solve p-MPUD instances. But the CPU time that *LR* and *BB* spent rose significantly with n. For p-MPUD, the MIP optimizer in Xpress-MP not only was unable to solve the linearization model (7) within the time limit of 10,800 s when n reached 50, but also ran out of memory and thus could not return a solution when $n = 100$, 150 and 200 (denoted by “n/a” in Tables 8 and 9). *BB* was unable to find an optimal solution within the time limit of 10,800 s when solving a p-MPUD instance with 50 or more nodes.

Tables 9 gives the hit rate (proportion of instances for which each algorithm returned the best known solution) and the average optimality gap (relative difference between the returned solution and the best known solution) for p-MPUD instances. Note that if *LR* or *BB* terminated when reaching the time limit, the solution it obtained might not be the true optimum. The impact of p was insignificant on solution quality of the algorithms. Therefore, the summary measures are organized in the table with respect to factors other than p.

For p-MPUD, *LR* seldom returned the best known solution within 3 h of CPU time once n was 50 or beyond. *BB* and *INT* seemed to return better solutions than *LR* in general for networks with 50 or more nodes. It is interesting to notice that *INT* was able to find the same solution as *BB* in every instance on a network with at least 50 nodes.

In view of the above computational results, we realize that the number of nodes on a network n is a major determinant of an algorithm's performance. Our computational experience not shown here suggests that other parameters such as the number of facilities to locate p, the risk attitude coefficient λ and the distributions of the mean and variance of the demand weights do not have a substantial effect on the CPU and solution quality of the algorithms except that the sign of λ significantly affects the CPU time taken by the MIP optimizer in Xpress MP to solve model (P_C). We recommend Glover's linearization method for solving the vertex p-MPUD on networks with no more than 25 nodes. The interchange heuristic and the branch and bound algorithm can be used to solve p-MPUD on other networks.

In network location problems n usually defines the problem size. The computational studies in the uncertain facility location literature seldom considered networks with 150 or more nodes even though networks in reality could be much larger. This restriction might be attributed to the intractability of uncertain location problems, many of which are NP-Hard in nature and would take very long time to solve by exact solution procedures. In this study we chose to solve instances on networks with 200 nodes at maximum to balance the applicability of the generated instances and the capability of the algorithms, in particular, the linearization method and the branch and bound algorithm. The linearization method requires a lot of decision variables for a huge network and the resulting models might not be solvable using an MIP optimizer due to insufficient memory. We note that Eq. (6) cannot ensure that nodes are assigned to the closest facility. As shown in the previous section, the solutions obtained with or without the closest assignment restriction may differ significantly. Therefore, the bounds of sub-problems in the branch and bound algorithm could be quite loose. This partly explains why the branch and bound algorithm was not very efficient for model (P_M).

Brimberg and Hodgson [5] noted that networks had become extremely large in real applications as the economy became globalized and hence it would be wise to apply heuristics that make a trade-off between solution quality and CPU time. In this study interchange heuristics were presented. We understand that heuristics could be developed in a more innovative way, which is a possible future research area.

7. Summary and outlook

In this paper we studied the median and center location problems from the point of view of a decision maker who has a mean–variance objective. We showed how the optimal facility locations depend on the risk attitude of the decision maker and provided some insights on factors that would affect the optimal locations.

For future research, one can extend the analysis for the median and center location problems in this study to more realistic cases such as the capacitated facility location problem [4]. We also note that it would be interesting to compare different risk measures for location analysis problems.

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Appendix

Solution process for Example 2.

$\mathbf{X}_2 = \{x_{(1)}, x_{(2)}\}$ denotes the locations of the facilities with $\delta = \{3 \le x_{(1)} \le 5;\ 0 \le x_{(2)} \le 2\}$. Note $d(1, \mathbf{X}_2) = \min(x_{(1)}, x_{(2)})$, $d(2, \mathbf{X}_2) = \min(5 - x_{(1)}, 5 - x_{(2)})$ and $d(3, \mathbf{X}_2) = 2 - x_{(2)}$. It follows that $\varpi(1, x_{(1)}) = \{x_{(1)} \le x_{(2)}\}$, $\varpi(1, x_{(2)}) = \{x_{(1)} \ge x_{(2)}\}$, $\varpi(2, x_{(1)}) = \{x_{(1)} \ge x_{(2)}\}$, and $\varpi(2, x_{(2)}) = \{x_{(1)} \le x_{(2)}\}$. As a result, we have $\Gamma = \{\gamma_1, \gamma_2\}$, which consists of two elements with $\gamma_1 = \{x_{(1)} \le x_{(2)}\}$ and $\gamma_2 = \{x_{(1)} \ge x_{(2)}\}$. Note that γ_1 and δ are conflicting and that γ_2 is implied by δ. We may consider the constraints in δ only, under which the objective function is expressed as

$$\begin{aligned} F(\mathbf{X}_2) = {} & 4x_{(2)} + 3(5 - x_{(1)}) + 5(2 - x_{(2)}) + \lambda[4x_{(2)}^2 + 4(5 - x_{(1)})^2 \\ & + 9(2 - x_{(2)})^2 - 4.8x_{(2)}(5 - x_{(1)}) - 6x_{(2)}(2 - x_{(2)}) \\ & + 3.6(5 - x_{(1)})(2 - x_{(2)})]. \end{aligned} \tag{10}$$

When $\lambda = 1$, the above expression reduces to

$$F(\mathbf{X}_2) = 4x_{(1)}^2 + 8.4x_{(1)}x_{(2)} + 19x_{(2)}^2 - 50.2x_{(1)} - 91x_{(2)} + 197.$$

According to Wolfe [36], an optimal solution to $F(\mathbf{X}_2)$ subject to the constraints in δ can be found by applying a revised simplex technique on the following linear programming model given by the Karush–Kuhn–Tucker conditions:

$$\begin{aligned} \min \quad & y_1 + y_2 + y_3 + y_4 + y_5 + y_6 \\ \text{s.t.} \quad & \\ & 8x_{(1)} + 8.4x_{(2)} + \lambda_1 - \lambda_2 + y_1 = 50.2 \\ & 8.4x_{(1)} + 38x_{(2)} + \lambda_3 - \lambda_4 + y_2 = 91 \end{aligned}$$

$$x_{(1)} + v_1 + y_3 = 5$$
$$x_{(1)} - v_2 + y_4 = 3$$
$$x_{(2)} + v_3 + y_5 = 2$$
$$-x_{(2)} + v_4 + y_6 = 0$$
$$\text{all variables } \geq 0, \tag{11}$$

where λ_j' s, v_j' s and y_j' s are Lagrangian multipliers, slack variables, and artificial variables, respectively, while the complementary slackness conditions $\lambda_j v_j = 0 \quad j = 1, 2, \ldots, 6$ are implicitly satisfied with a restricted basis entry rule. Solving model (11) we have $\mathbf{X}_2 = \{x_{(1)} = 4.897,\ x_{(2)} = 1.312\}$ as the optimal solution with $F(\mathbf{X}_2) = 14.375$.

If $\lambda = -1$, Eq. (10) can be re-written as

$$F(\mathbf{X}_2) = -4x_{(1)}^2 - 8.4x_{(1)}x_{(2)} - 19x_{(2)}^2 + 50.2x_{(1)} + 89x_{(2)} - 147.$$

Applying the global optimization algorithm in Kalantari and Rosen [21], we obtain the optimal solution $\mathbf{X}_2 = \{x_{(1)} = 3, x_{(2)} = 0\}$ with $F(\mathbf{X}_2) = 50.4$.

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